A novel method based on signal sparsity to obtain fractional sample delay

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***Abstract*—In many applications, such as communication, speech coding, and audio technology, not only the sampling frequency but also the actual sampling instants are of crucial importance. Fractional sample delay has been widely used in the domain of signal processing. There are many methods to obtain higher delay resolution. However the common defect is that they can not provide a continuous variable accurate delay. In this paper we take into account the characteristic of the information carried by the signal, which is the essence of the signal. A novel method based on the signal sparsity is prosed. Compared to the traditional interpolation algorithm, this method has lower normalized mean square error (NMSE) and can obtain arbitrary sample delay.**

***Index Terms*—fractional sample delay, signal sparsity, interpolation algorithm**

I. Introduction

The sampling rate must satisfy the Nyquist theorem in order to represent adequately the original continuous signal in traditional signal processing. However, the appropriate sampling rate alone is not sufficient for many applications, the sampling instants must be properly selected too. Because some values which lie somewhere between two samples are always needed, signal delay is widely used in many signal processing applications. It is easily to implement signal delay for analog signals. However, it becomes more difficult for digital signals. Furthermore, there is always a need for an accurate delay which is a fraction of the sampling period in the actual engineering application, including digital communication [1], digital beam steering [2], sample rate conversion [3], and speech processing [4].

The literature of fractional sample delay includes a wide variety of design methods, a number of which have been discussed in the introductory reference [4] and [5]. The problem is viewed mainly as a time domain interpolation problem. The simplest method is modification of standard sampling rate, which will lead to higher computational complexity and hardware consumption. The Lagrange interpolation for timing adjustment in digital modems has been proposed in [6] and truncated $sinc$ interpolation in symbol synchronization has been considered in [7]. This two methods can be easily implemented and decrease the processing data volume comparing with the dense sampling but have low delay resolution. There are also many design methods for interpolation filters using Finite Impulse Response (FIR) and Infinite Impulse Response (IIR) filters [10], [11], [12]. They have high delay resolutions but have large calculation complexity. Each of all the above designs has its own advantage which makes it suitable to be deployed in some special classes of applications. However a common drawback of them is that they did not consider the characteristics of the signal itself. In this paper we consider the characteristics of the information carried by the signal and a novel method based on the signal sparsity is proposed.

Sparsity is the signal structure behind many compression algorithms that employ transform coding. As is known, most of the natural signals are sparse and can be accurately expressed by exploiting the information sparsity feature. It has a rich history of application utilization in signal processing problems in the last century, particularly in imaging, including denoising, deconvolution, restoration and inplanting [8], [9]. In this paper, a novel method based on this feature is proposed to achieve arbitrary fractional sample delay. Sparsity makes the signal need less resources for the transmission and storage. Because the original time continuous signal is a linear combination of the sparse basis composed of a series continuous functions, the operations for the signal can be transferred into the operations for the sparse basis. Then the fractional sample delay for the corresponding digital signal is translated into the delay and sampling of the sparse basis in the analog domain.

The paper is organized as follows. Section II presents the theoretical analysis of the fractional sample delay system. The new method proposed in the paper will be introduced in detail in section III. Finally, we will give out the simulation results in section IV and have a conclusion in section V.

II. Theoretical Analysis Of The Fractional Sample Delay System

For a time continuous signal $x(t)$, delaying an amount t_d is conceptually simple (Fig. 1a). A time continuous ideal delay can be defined as a linear operator, which yields its output $y(t)$ as

$$y(t) = x(t - t_d). \tag{1}$$

However it is more difficult for an uniformly sampled baseband digital signal. Sampling the continuous signal $x(t)$ at time instants $t = nT$, where T is the sampling period, the digital signal can be expressed as $\hat{x}(n) = x(nT)$. With delaying an interval D which is a positive real number that can be split into the integer part $int(D)$ and the fractional part $d(0 \leq d < 1)$ as $D = int(D) + d$, the output $\hat{y}(n)$ can be written as

$$\hat{y}(n) = \hat{x}(n - D). \tag{2}$$

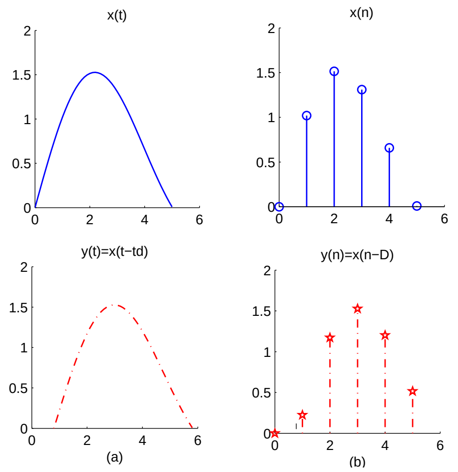


Fig. 1: Delaying of (a) continuous time signal (b) discrete time signal

However, (2) is meaningful only for integer values of D, where the output sample is one of the previous signal samples. But for non-integer values of D, the output value would lie somewhere between two samples (Fig. 1b).

The linear system which calculate one output sample $\hat{y}(n)$ using a set of adjacent input samples $\hat{x}(n)$ and the fractional interval D is usually called interpolation filter. The output of interpolation filter is given by

$$\hat{y}(n) = h(n) * \hat{x}(n), \tag{3}$$

where $h(n)$ is the impulse response of the interpolation filter and $*$ indicates the convolution operation. Ideally, $h(n)$ is a shifted and sampled $sinc$ function, that is, $h(n) = sinc(n - D)$. The impulse response is infinite length, then the ideal fractional sample delay filter is noncausal and non-realizable. To produce realizable filter, some finite length and causal approximation for the ideal $sinc$ function has to be used.

The two usual approaches to design casual FIR filters are windowed $sinc$ function and maximally-flat FIR approximation. Lagrange interpolation is obtained as a maximally-flat FIR approximation of fractional sample delay filter. Its coefficients are given by the following equation:

$$h(n) = \prod_{k=2, k\neq n}^{N} \frac{D-k}{n-k}, n = 0, 1, \cdots, N \tag{4}$$

where N is the order of the filter. The coefficients of the FIR filter also can be obtained using windowing method as

$$h(n) = w(n - D) sinc(n - D), \tag{5}$$

where $w(n - D)$ is a length-$N + 1$ window sequence shifted by the appropriate delay value D.

In case of IIR filters, there still are many design methods but the error curves are asymmetric and their stability must be taken into account.

III. Fractional Sample Delay Based On Signal Sparsity

Fractional sample delay filters yield the best approximation when the total delay D is close to $N/2$ for FIR filters and close to N for IIR filters. In order to get better performance, not only the coefficients of the filters even the order of the filters need to be changed as the delay D varies, which will cause large calculation complexity. To conquer the drawback, a novel algorithm based on signal sparsity is introduced. This method can achieve arbitrary fractional sample delay and has higher delay resolution.

A. *Signal sparsity*

Signal sparsity indicates the information carried by the signal and is the most essential characteristic of the signal. To introduce the notion of sparsity, we rely on a signal representation in a given basis $\{\psi_m\}_{m=1}^{M}$ for $\mathbb{R}^M$. Every signal $x \in \mathbb{R}^M$ is an M-point real-valued discrete-time signal. Then x is representable in terms of M coefficients $\{\theta_m\}_{m=1}^{M}$ as

$$x = \sum_{m=1}^{M} \theta_m \psi_m = \mathbf{\Psi}\Theta, \tag{6}$$

where $\mathbf{\Psi}$ is an $M \times M$ matrix using ψ_m as columns and Θ is an $M \times 1$ coefficient vector composed by the coefficients θ_m. Similarly, if we use a matrix $\mathbf{\Psi}$ containing M column vectors of length L with $L < M$, then for any vector $y \in \mathbb{R}^L$ it can be expressed with $\Theta \in \mathbb{R}^M$ as $y = \mathbf{\Psi}\Theta$. We refer to $\mathbf{\Psi}$ as the sparse dictionary.

If the coefficient vector $\Theta \in \mathbb{R}^M$ with only $K(K << M)$ entries are nonzero, we can say that the signal x is K-sparse. Although most of the natural signals are not sparse in a strict sense, they always have concise representations in a proper sparse basis or dictionary, in the sense that the stored magnitudes of sparse coefficients decay quickly. Sparsity determines the efficiency of signal acquisition and decreases the needed resource for storage and transmission.

B. *Prosed method based on signal sparsity*

The framework of the novel method based on signal sparsity is described in Fig. 2. The core algorithm to obtain fractional sample delay is within the dashed box. It mainly has three steps. First, the suitable sparse basis $\mathbf{\Psi}(t)$ is constructed according to the characteristics of the original analog signal $x(t)$. In this paper, $\mathbf{\Psi}(t)$ is assumed to be known. Then the coefficient vector $\mathbf{\Theta}$ is gotten through solving l_1-norm optimization. Finally, the output signal $\hat{y}(n) = \hat{x}(n - D)$ with delaying an interval D is obtained by linear operation.

Extend the signal sparsity from the digital domain to the analog domain. Supposing the time continuous signal $x(t)$ has finite information rate, it is reasonable to assume that it can be represented or approximated as a linear combination of a finite number of parameters in sparse basis or dictionary which contains a series continuous functions $\{\psi_m(t)\}_{m=1}^{M}$. Hence,

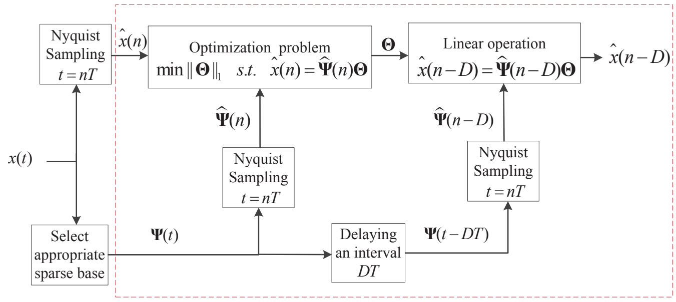


Fig. 2: The framework of the novel method based on signal sparsity

the signal $x(t)$ can be mathematically expressed as

$$x(t) = \mathbf{\Psi}(t)\mathbf{\Theta} = \sum_{m=1}^{M} \theta_m \psi_m(t), \tag{7}$$

with $\theta_m, t \in \mathbb{R}$, where $\mathbf{\Psi}(t) = [\psi_1(t), \cdots, \psi_M(t)]$, $\mathbf{\Theta} = [\theta_1, \cdots, \theta_M]^T$ is a vector composed of coefficients $\{\theta_m\}_{m=1}^{M}$. In case where there are a small number of nonzero or large magnitude entries in $\mathbf{\Theta}$, the analog signal $x(t)$ is sparse or compressible in basis $\mathbf{\Psi}(t)$ which maps the discrete vector of coefficients onto a continuous signal. Although each of the basis element $\psi_n(t)$ may have high bandwidth, the signal itself has finite information freedom.

Eq.(7) shows that $x(t)$ is a linear combination of $\psi_m(t)$ and coefficients θ_m are constants which are irrelevant to time when the sparse basis $\mathbf{\Psi}(t)$ is determined. So sampling the continuous signal $x(t)$ at time instants $t = nT$, the digital signal $\hat{x}(n)$ can be rewritten as

$$\begin{aligned} \hat{x}(n) &= x(nT) \\ &= \sum_{m=1}^{M} \theta_m \psi_m(nT) \\ &= \hat{\mathbf{\Psi}}(n)\mathbf{\Theta}, \end{aligned} \tag{8}$$

where $\hat{\mathbf{\Psi}}(n)$ is the discretization of $\mathbf{\Psi}(t)$ and can be expressed as $\hat{\mathbf{\Psi}}(n) = \mathbf{\Psi}(nT) = [\psi_1(nT), \cdots, \psi_M(nT)]$.

The coefficient vector $\mathbf{\Theta}$ can be acquired by solving the following question

$$\arg\min \parallel \mathbf{\Theta} \parallel_1 \quad s.t. \quad \hat{x}(n) = \hat{\mathbf{\Psi}}(n)\mathbf{\Theta}. \tag{9}$$

This convex optimization problem, also known as Basis Pursuit (BP) [13], can be simplified to a traditional linear programming problem. An alternative to the optimization-based approach is greedy algorithm based on dynamic programming, such as Matching Pursuit (MP) [14] and Orthogonal Matching Pursuit (OMP) [15]. In this paper, OMP algorithm is adopted to obtain the coefficient vector for its lower computation complexity. A short algorithmic description of OMP is listed in Table I.

According to (8), as soon as $\mathbf{\Theta}$ is gotten, the output $\hat{y}(n)$ with delaying an interval D can be obtained by

TABLE I: The short description of OMP algorithm

Initialization	set non-zero elements $\mathbf{I}$ as empty, set residual error as $\mathbf{r} = \mathbf{x}$.
Iteration	while halting criterion false, do
	(1) Correlate all columns of $\mathbf{\Psi}$ with $\mathbf{r}$, choose the largest element by magnitude and add its index to $\mathbf{I}$: $i = \arg\max_i \lvert \mathbf{\Psi}_i' * \mathbf{r} \rvert, \mathbf{I} = \mathbf{I} \bigcup \{i\}$.
	(2) Find an estimate $\widehat{\mathbf{\Theta}}$ that minimizes $\lVert \mathbf{x} - \widehat{\mathbf{\Psi}}\widehat{\mathbf{\Theta}} \rVert$: $\widehat{\mathbf{\Theta}} = \arg\min_{\widehat{\mathbf{\Theta}}} \lVert \mathbf{x} - \widehat{\mathbf{\Psi}}\widehat{\mathbf{\Theta}} \rVert$, where $\widehat{\mathbf{\Psi}}$ is obtained from the columns in $\mathbf{\Psi}$ having indices in the set $\mathbf{I}$.
	(3) Update the residual:$\mathbf{r} = \mathbf{x} - \widehat{\mathbf{\Psi}}\widehat{\mathbf{\Theta}}$.
	end while
Output	The coefficient vector $\mathbf{\Theta} = \widehat{\mathbf{\Theta}}$.

$$\begin{aligned} \hat{y}(n) &= \hat{x}(n-D) \\ &= \hat{\mathbf{\Psi}}(n-D)\mathbf{\Theta} \\ &= \sum_{m=1}^{M} \theta_m \psi_m((n-D)T) \\ &= \sum_{m=1}^{M} \theta_m \psi_m(t-DT)|_{t=nT}. \end{aligned} \tag{10}$$

Because $\mathbf{\Psi}(t)$ is composed by a series continuous functions $\{\psi_m(t)\}_{m=1}^{M}$, $\hat{\mathbf{\Psi}}(n-D) = \mathbf{\Psi}((n-D)T)$ can be easily obtained through delay and sampling of $\mathbf{\Psi}(t)$: First, $\mathbf{\Psi}(t)$ delays an interval DT and $\mathbf{\Psi}(t-DT)$ is achieved. Then sampling $\mathbf{\Psi}(t-DT)$ at $t = NT$ and $\mathbf{\Psi}((n-D)T)$ is obtained. The fractional sample delay for digital signal in digital domain is translated into the delay and sampling of the sparse basis in the analog domain.

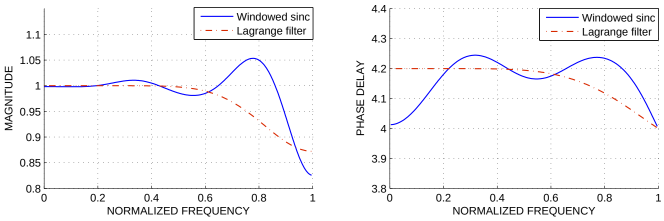
It is shown from (10) that coefficients θ_m are independent with the interval value D, then the coefficients do not need to be calculate again when interval D varies. Compared with the traditional filter approximations, this method not only easily implemented but also can realize a continuous variable accurate delay.

IV. Simulation Results

In this section, we show the performance gain realized by employing the proposed method and compare it with the traditional interpolation algorithms. In simulations, the time continuous signal $x(t)$ is expressed as

$$x(t) = \sum_{i=1}^{4} a_i cos(2\pi f_i t + \tau_i), \tag{11}$$

where the amplitude coefficients are $a_i = \{0.4, 0.6, 0.8, 1\}$, the carriers are $f_i = \{5MHz, 8MHz, 16MHz, 20MHz\}$ and the delays are $\tau_i = \{0.5, 0.2, 0.1, 0\}$. The sampling frequency is $F_s = 50MHz$, then the digital signal $\hat{x}(n)$ is described as $\hat{x}(n) = x(nT_s)$, where $T_s = 1/F_s$ is the sampling period. The



(a) (b)

Fig. 3: Simulation results

normalized mean square error (NMSE) is calculated to reflect the performance and defined as

$$NMSE = \frac{\|\widetilde{y}(n) - \hat{y}(n)\|^2}{\|\hat{y}(n)\|^2}, \tag{12}$$

where $\widetilde{y}_n$ is the estimated value.

Fig. 3 shows the magnitude and phase delay responses of two FIR filters where the interval value is $D = 4.2$ and the order of filter is $N = 8$. FIR filters use finite length to approximate infinite length impulse response, which introduces a truncated error. It is also shown from Fig. 3 that the responses of windowed $sinc$ filter and Lagrange filter are not ideal and there is an error. The NMSE for these two FIR approximations and the proposed method under different signal to noise ratio (SNR) is shown in Fig. 4. Windowed $sinc$ filter and Lagrange filter have similar performances. However the proposed method has much lower NMSE and a better performance than the traditional FIR approximations.

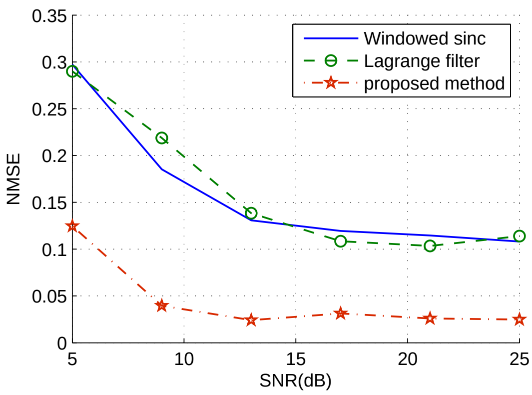


Fig. 4: NMSEs for three methods under different SNRs

NMSE for three methods under different fractional intervals are displayed in Fig. 5, where the SNR is $20dB$ and the interval value is $4 < D < 5$. In order to get good performance, the order of FIR filters is respectively $8, 9, 10$ when $4 < D < 4.25, 4.25 \leq D < 4.75, 4.75 \leq D < 5$. It is shown that NMSE changes tremendously with fractional interval varies when using FIR approximations, but it is almost the same for the proposed method. Furthermore the proposed method has much lower NMSE.

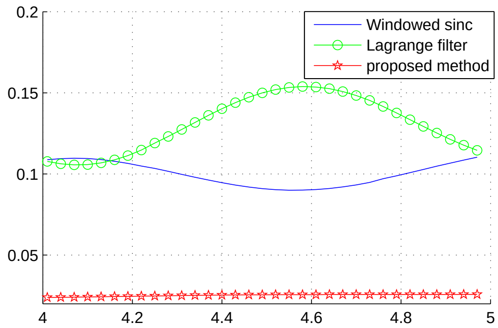


Fig. 5: NMSEs for three methods under different fractional intervals

Fig. 4 and Fig. 5 show that the proposed method has better performance than the traditional FIR filter approximations. The FIR filter approximations makes the impulse response from infinite-length into finite-length, which introduces a truncated error. In order to get good performance, the order of FIR filters and the coefficients vary with the interval value, which leads large amount of calculation. However the proposed method not only avoids the artificial truncation error and considers the nature of the signal, and has a better performance.

V. Conclusion

Filter approximations are usually used to obtain fractional sample delay. However these methods not only have large amount of calculation but also can not achieve a continuous delay. In order to implement fractional sample delay with more accurate resolution, we take into account the characteristics of the information carried by the signal and propose a novel method based on the signal sparsity. By applying the sparsity, the fractional sample delay of the digital signal is transferred into the delay and sampling operations of the continuous sparse basis of the original analog signal. Compared with the traditional FIR approximations, this method not only has much better performance but also can provide a continuous variable accurate delay only at expense of initial computational complexity to get the coefficient vector.

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References

[1] Candan, C.: "An efficient filtering structure for Lagrange interpolation",IEEE Signal Process. Lett., 2007, vol. 14, no. 1, pp. 17-19.

[2] Murphy, P., Krukowski, A., and Tarczynski, A., "An efficient fractional sample delayer for digital beam steering". Proc. 1997 IEEE Int. Conf.Acoustics, Speech, Signal Processing, (ICASSP), Munich, Germany,April 1997, Vol. 3, pp. 2245-2248.

[3] Rajamani, K., Lai, Y.-S., and Farrow, C.W., "An efficient algorithm for sample rate conversion from CD to DAT", IEEE Signal Process. Lett., 2000, vol. 7, no. 10, pp. 288-290.

[4] Laakso, T.I., Välimäki, V., Karjalainen, M., and Laine, U.K., “Splitting the unit delay C tools for fractional delay filter design”, IEEE Signal Process. Mag., 1996, vol. 13, no. 1, pp. 30-60.
[5] Välimäki, V., and Laakso, T.I., “Principles of fractional delay filters”.Proc. 2000 IEEE Int. Conf. Acoustics, Speech, Signal Processing,(ICASSP), Istanbul, Turkey, June 2000, Vol. 6, pp. 3870-3873.
[6] G.-S. Liu1 and C.-H. Wei, “Programmable fractional sample delay filter with Lagrange interpolation”, Electron. Letter, vol. 26, pp. 1608-1610, Sep. 1990.
[7] J. Armstrong and D. Strickland, “Symbol synchronization using signal samples and interpolation,” IEEE Trans.Commun., vol. 41, pp. 318-321, Feb. 1993.
[8] I. F. Gorodnitsky and B. D. Rao, “Sparse signal reconstruction from limited data using FOCUSS: A reweighted minimum norm algorithm,” IEEE Trans. Signal Process., vol. 45, no. 3, pp. 600-616, Mar. 1997.
[9] A. M. Bruckstein, D. L. Donoho, and M. Elad, “From sparse solutions of systems of equations to sparse modeling of signals and images,” SIAM Rev., vol. 51, no. 1, pp. 34-81, Feb. 2009.
[10] Shailey Minocha, S.C. Dutta Roy, Balbir Kumar, “A Note On The FIR Approximation of a Fractional Sample Delay,” International Journal of Circuit Theory and Applications, Vol. 21, 1993, pp. 265-274.
[11] Peter J. Kootsookos and Robert C. Williamson, “FIR Approximation of Fractional Sample Delay Systems,” IEEE Transactions on Circuits and Systems-U: Analog and Digital Signal Processing Vol. 43, No. 3, 1996, pp. 269-271.
[12] G. D. Cain, N.P. Murphy and A. Tarczynski, “Evaluation of Several Variable FIR Fractional Sample Delay Filters,” Proceedings of the International Conference on Acoustics Speech and Signal Processing, Tokyo, Japan, pp. 32-29, 1994.
[13] S. Chen, D. Donoho, M. Saunders, “Atomic decomposition by basis pursuit,” SIAM Journal on Scientific Computing, vol. 20, no. 1, pp. 33-61, 1998.
[14] R. Nef, A. Zakhor, “Very low bit rate video coding based on matching pursuits,” IEEE Transaction on Circuits and Systems for Video Technology, vol. 7, no. 1, pp. 158-171, 1997.
[15] Y. Pati, R. Rezaiifar, and P. Krishnaprasad, “Orthogonal matching pursuit: recursive function approximation with applications to wavelet decomposition,” in Proc. Asilomar Conference on Signals, Systems and Computers, vol. 1, Nov. 1993, pp. 40-44.